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LONG TERM PRECESSION MODEL

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Abstract. Precession is the secular and long-periodic component of the motion of the Earth's spin axis in space, exhibiting a motion of about $50''$/year around the pole of the ecliptic. All precession models, used in astronomy so far (Newcomb, IAU 1976, IAU 2006) approximate this motion by polynomial expansions of time. These models are however valid, with very high accuracy, only in the close vicinity of the reference epoch J2000.0. For epochs that are more distant (several centuries), this approximation quickly deviates from reality. As a reaction to this problem, a new model, comprising very long-period terms fitted to a numerical integration of the motion of solar system bodies, has recently been developed by the present author in cooperation with N. Capitaine (France) and P. Wallace (United Kingdom) and published in the European journal Astronomy and Astrophysics. A shortened description of the new model, including an evaluation of its accuracy, is presented.

1. INTRODUCTION

The transformation between the terrestrial and celestial reference frame is given by five Earth Orientation Parameters (see Fig. 1). They define the position of the spin axis in the Earth's body (polar motion), the angle of proper rotation (Universal Time) and the position of the spin axis in space (precession-nutation). The position of the axis of rotation of the Earth exhibits, under the dominant influence of the Moon and the Sun, a rather complicated motion in space. Its very long-periodic part, precession, is the slow motion of the pole of Earth's rotation P around the pole of the ecliptic C. The angle between the two poles (obliquity) is approximately constant, today roughly equal to 23.5°. Precession was known already to Hipparchos, since it causes the growth of ecliptical longitudes of the stars by about $50''$ per year; the spin axis makes one revolution in about 26 thousand years (Platonic year). This motion is in reality rather complicated: the pole of the ecliptic itself is not fixed with respect to the stars – it exhibits precession of the ecliptic (formerly called planetary precession). It is dominantly caused by the attractive forces of all bodies of the solar system on the motion of the Earth around the barycenter of the solar system. The axis of rotation of the Earth exhibits a motion around the moving pole of ecliptic under the torques exerted by the Moon, Sun, and planets on the rotating oblate Earth, precession of

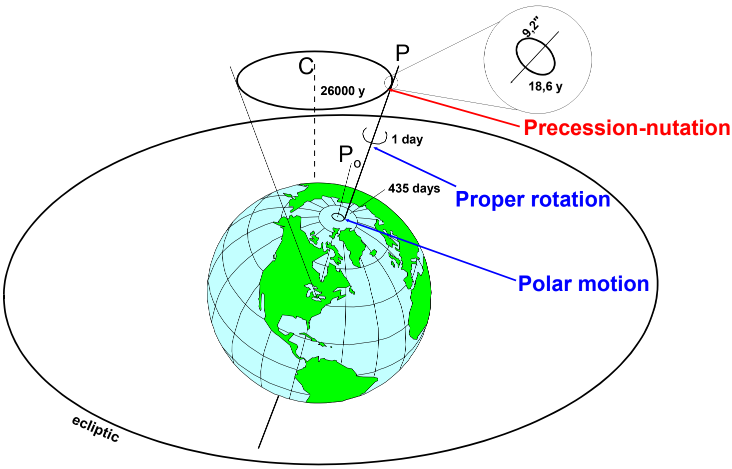

Figure 1: Earth Orientation Parameters.

the equator (formerly luni-solar precession). Neither the obliquity, nor the rate of precession are strictly constant.

All precession models used so far are expressed in terms of polynomial development of time. The most recent model IAU 2006 (Capitaine et al. 2003) is very accurate, but usable only for a limited time interval (several centuries around the epoch J2000); its errors rapidly increase with longer time spans. In reality, precession represents a very long-periodic process, whose periods reach hundreds of centuries. This can be demonstrated by comparison with the numerically integrated equations of motion of the Earth in the solar system and its rotation (Vondrák et al. 2009, 2011a). Fig. 2 (here reproduced from paper by Vondrák et al. 2009) displays the motion of the axis of rotation of the Earth during about 1.5 precession cycles, as given by long-term numerical integration (LT integration) and different analytical models – Lieske et al. (1977), Simon et al. (1994), and two models by Capitaine et al. (2003): one computed from the expansions of precession angles ζ_A, θ_A, and one from the expansions of direction cosines X_A, Y_A. The position of the axis of rotation at the basic epoch J2000.0 is the point $X = Y = 0$, pole of the ecliptic is located in the center of the figure. The models are not graphically distinguishable in the interval $\pm$50cy around J2000, but they start to differ significantly outside the interval $\pm$100cy.

We assume that precession includes only periods longer than 100 centuries; shorter ones represent nutation. Our aim was to find relatively simple expressions of different precession parameters, with accuracy comparable to the IAU 2006 model near the epoch J2000.0, and lower, slowly degrading accuracy outside the interval $\pm$1000 years (up to several minutes of arc at the extreme epochs $\pm$200 thousand years). The paper describing the new model in detail has recently been published by Vondrák et al. (2011b). Below is given an abridged description of the model, followed by a new assessment of its accuracy and comparison with other models.

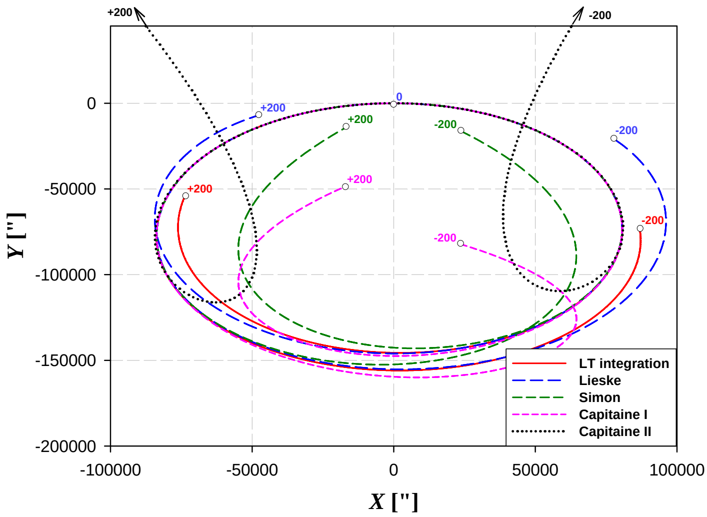

Figure 2: Different models of precession in the interval ±200cy around J2000.0, and comparison with numerical integration.

2. NUMERICAL INTEGRATION, LONG TERM EXPRESSIONS

We used the numerically integrated values of the following four parameters

- the precession of the ecliptic $P_A = \sin\pi_A \sin\Pi_A, Q_A = \sin\pi_A \cos\Pi_A$, calculated with the Mercury 6 package by Chambers (1999), considering only the eight 'classical' planets, and
- the general precession/obliquity p_A, ε_A, computed by Laskar et al. (1993),

to calculate time series for all other precession parameters in the interval ±200 thousand years from J2000.0, with 100-year steps. The numerical integrations depend on the initial conditions that are based on observations. In the first case, these are the optical and radar observations of the planets, in the second one the observations of the Earth orientation parameters by VLBI. Namely the latter led to small corrections that we had to apply to Laskar's values of p_A, ϵ_A (due to a slightly different value of dynamical ellipticity of the Earth, the rate of change of the dynamical form factor J_2, planetary tilt effects and the tidal effects, neglected in Laskar's solution).

To estimate the accuracy of the numerical integrations above, we tested them against the values obtained independently and found that the differences from other solutions at both extreme epochs do not exceed the level of 20 arcseconds. The neglected perturbations by asteroids have recently been shown by Aljabaae and Souchay (2012) to be very small - peak to peak quasi-periodic effects in Earth's inclination are smaller than $0.05''$, the periods are typically shorter than 100 years. Thus we concluded that the accuracy of the numerical integration, including both numerical errors and imperfections of the models used, is sufficient for our purpose.

The central part of the data (±1000 years from the epoch J2000.0) was replaced by IAU 2006 values to make the new model consistent with the recent model accepted by

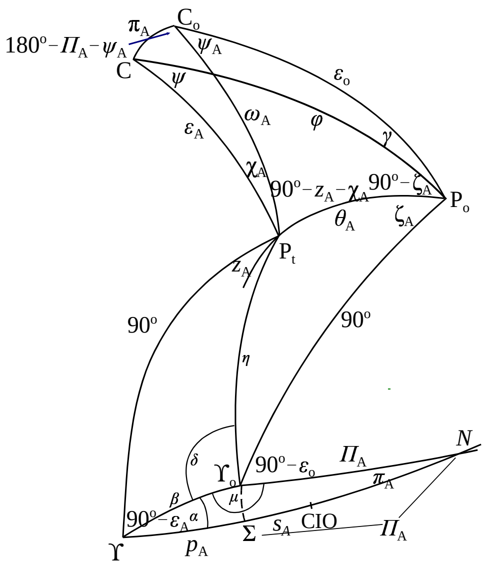

Figure 3: Precession parameters.

the IAU. From the values of the precession parameters P_A, Q_A, p_A and ϵ_A, different precession parameters were calculated in the interval ± 200 millennia from J2000.0 with 100-year steps, solving several spherical triangles depicted in Fig. 3, in which C$_\circ$ and C denote the positions of the pole of ecliptic at the epochs J2000.0 and T, respectively, P$_\circ$, P$_t$ are the poles of rotation of the Earth and $\Upsilon_\circ$, Υ vernal points at the same epochs, and CIO stands for Celestial Intermediate Origin.

We proceeded from the bottom of the figure upwards and obtained first the auxiliary angles α, β, μ from the spherical triangle $\Upsilon\Upsilon_\circ$N, then the angles η, δ by solving the triangle $\Upsilon\Upsilon_\circ$P$_t$, and, from triangle $\Upsilon_\circ$P$_t$P$_\circ$, we got the precession angles θ_A, ζ_A. From the triangle P$_\circ$P$_t$C$_\circ$ then followed the precession parameters ω_A, ψ_A and from the triangles P$_t$CC$_\circ$, P$_\circ$P$_t$C$_\circ$ the parameters χ_A, z_A.

Instead of precession angles θ_A, z_A, ζ_A we used direction cosines $X_A = \sin\theta_A \cos\zeta_A$, $Y_A = -\sin\theta_A \sin\zeta_A$, $V_A = \sin\theta_A \sin z_A$, $W_A = \sin\theta_A \cos z_A$; the angles θ_A, ζ_A and z_A exhibit large discontinuities (of about 94° for θ_A, 180° for ζ_A and z_A) at irregular intervals: there is also a change of sign approximately each 26,000 years. This makes the long-term analytical approximation of these precession angles extremely difficult, while the direction cosines are continuous.

The time series of all parameters calculated above were then approximated by a cubic polynomial plus up to 14 long-periodic terms of the general form (T is the time in centuries from J2000.0, P_i is the period and n the number of periodic terms)

$$a + bT + cT^2 + dT^3 + \sum_{i=1}^{n} \left(C_i \cos 2\pi T/P_i + S_i \sin 2\pi T/P_i \right), \tag{1}$$

so that the fit is best around J2000.0. This was assured by choosing appropriate weights (equal to 10^4 in the central part and to $1/T^2$ outside this interval). The periods were found beforehand using the Vaníček's method (Vaníček 1969), based on the least-squares method, as modified by Vondrák (1977), and verified with the ones found by Laskar et al. (1993, 2004) from much longer time series. Weighted least-squares estimation (with fixed values of the periods) was then used to determine the cosine/sine amplitudes of individual periodic terms.

We derived the long-term expressions of the following precession parameters (some of them being precession angles, some direction cosines, expressed in terms of certain precession angles):

- precession angles: $p_A, \varepsilon_A, \omega_A, \psi_A, \chi_A, \varphi, \gamma, \psi$;
- direction cosines: $P_A = \sin\pi_A \sin\Pi_A$, $Q_A = \sin\pi_A \cos\Pi_A$, $X_A = \sin\theta_A \cos\zeta_A$, $Y_A = \sin\theta_A \sin\zeta_A$, $V_A = \sin\theta_A \sin z_A$, $W_A = \sin\theta_A \cos z_A$.

We also derived the expression for the CIO locator (the part that is due to precession), the small angular distance between the points Σ and CIO, s_A. All these angles are depicted in Fig. 3.

3. EXAMPLES

We present here, as typical examples, the long-term expressions of direction cosines of the pole of the ecliptic C, P_A, Q_A, and of the Earth's spin axis P_t, X_A, Y_A (both expressed in arcseconds):

The long-term expressions for the precession of the ecliptic are given as

$$\begin{aligned} P_A &= 5851.607687 - 0.1189000T - 0.00028913T^2 + 101 \times 10^{-9}T^3 + \textstyle\sum_P, \\ Q_A &= -1600.886300 + 1.1689818T - 0.00000020T^2 - 437 \times 10^{-9}T^3 + \textstyle\sum_Q, \end{aligned} \quad (2)$$

where the cosine/sine amplitudes of the periodic parts $\sum_P$, $\sum_Q$ are given in Tab. 1. Names of some of the terms in column 1 come from Laskar et al. (1993, 2004). The comparison of the long-term model of the precession of the ecliptic, P_A (top), Q_A

Table 1: Periodic terms in P_A, Q_A

term	C/S	$P_A['']$	$Q_A['']$	P[cy]
σ_3	C_1	-5486.751211	-684.661560	708.15
	S_1	667.666730	-5523.863691	
$-s_1$	C_2	-17.127623	2446.283880	2309.00
	S_2	-2354.886252	-549.747450	
	C_3	-617.517403	399.671049	1620.00
	S_3	-428.152441	-310.998056	
$-s_6$	C_4	413.442940	-356.652376	492.20
	S_4	376.202861	421.535876	
	C_5	78.614193	-186.387003	1183.00
	S_5	184.778874	-36.776172	
	C_6	-180.732815	-316.800070	622.00
	S_6	335.321713	-145.278396	
	C_7	-87.676083	198.296701	882.00
	S_7	-185.138669	-34.744450	
	C_8	46.140315	101.135679	547.00
	S_8	-120.972830	22.885731	

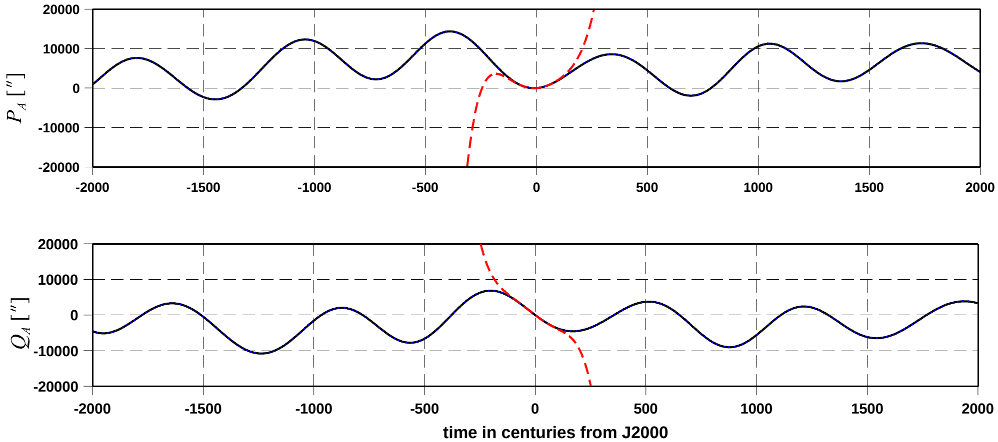

Figure 4: Long-term model of precession parameters P_A, Q_A – new model (dotted), integrated values (solid), and IAU 2006 (dashed).

(bottom) with integrated values and the IAU 2006 model is depicted in Fig. 4. The model and integrated values are so close that they are graphically indistinguishable. One can readily see that the expressions for P_A, Q_A of IAU 2006 model quickly deviate from the former ones. The pole of the ecliptic roughly describes a clockwise circular motion with the amplitude of about 1.5° and period of 71 millenia.

The expressions for the precession of the equator are

$$\begin{aligned} X_A &= 5453.282155 + 0.4252841T - 0.00037173T^2 - 152 \times 10^{-9}T^3 + \textstyle\sum_X, \\ Y_A &= -73750.930350 - 0.7675452T - 0.00018725T^2 + 231 \times 10^{-9}T^3 + \textstyle\sum_Y, \end{aligned} \quad (3)$$

where the cosine/sine amplitudes of the periodic parts $\sum_X$, $\sum_Y$ are displayed in Tab. 2. The comparisons of the long-term models of precession angles X_A(top) and Y_A (bottom) are shown in Fig. 5. Again, the model is graphically indistinguishable from the numeral integration. The pole of rotation describes a clockwise motion around the pole of the ecliptic, once per 26 millenia, its radius quasi-periodically changes between 22.5° and 24.2°, with period of about 71 millenia. The speed of the motion (general precession) is also not constant. The behavior of other precession parameters is similar to these.

4. PARAMETRIZATION OF PRECESSION MATRIX

Different combinations of the precession angles derived above can be used to compute precession matrix $\mathbf{P}$, necessary to transform coordinates of celestial bodies from the fundamental epoch J2000.0 to any epoch T:

- 'Lieske' parametrization (Lieske et al. 1977):
  $\mathbf{P} = \mathbf{R}_3(-z_A).\mathbf{R}_2(\theta_A).\mathbf{R}_3(-\zeta_A)$,

Table 2: Periodic terms in X_A, Y_A

term	C/S	$X_A['']$	$Y_A['']$	P[cy]
p	C_1	-819.940624	75004.344875	256.75
	S_1	81491.287984	1558.515853	
$-\sigma_3$	C_2	-8444.676815	624.033993	708.15
	S_2	787.163481	7774.939698	
$p - g_2 + g_5$	C_3	2600.009459	1251.136893	274.20
	S_3	1251.296102	-2219.534038	
$p + g_2 - g_5$	C_4	2755.175630	-1102.212834	241.45
	S_4	-1257.950837	-2523.969396	
$-s_1$	C_5	-167.659835	-2660.664980	2309.00
	S_5	-2966.799730	247.850422	
$-s_6$	C_6	871.855056	699.291817	492.20
	S_6	639.744522	-846.485643	
$p + s_4$	C_7	44.769698	153.167220	396.10
	S_7	131.600209	-1393.124055	
$p + s_1$	C_8	-512.313065	-950.865637	288.90
	S_8	-445.040117	368.526116	
$p - s_1$	C_9	-819.415595	499.754645	231.10
	S_9	584.522874	749.045012	
	C_{10}	-538.071099	-145.188210	1610.00
	S_{10}	-89.756563	444.704518	
	C_{11}	-189.793622	558.116553	620.00
	S_{11}	524.429630	235.934465	
$2p + s_3$	C_{12}	-402.922932	-23.923029	157.87
	S_{12}	-13.549067	374.049623	
	C_{13}	179.516345	-165.405086	220.30
	S_{13}	-210.157124	-171.330180	
	C_{14}	-9.814756	9.344131	1200.00
	S_{14}	-44.919798	-22.899655	

- 'Capitaine' parametrization (Capitaine et al. 2003):
  $\mathbf{P} = \mathbf{R}_3(\chi_A).\mathbf{R}_1(-\omega_A).\mathbf{R}_3(-\psi_A).\mathbf{R}_1(\varepsilon_\circ)$,
- 'Williams-Fukushima' parametrization (Fukushima 2003):
  $\mathbf{P} = \mathbf{R}_1(-\varepsilon_A).\mathbf{R}_3(-\psi).\mathbf{R}_1(\varphi).\mathbf{R}_3(\gamma)$,

in which $\mathbf{R}_i(\alpha)$ denotes the rotation matrix around $i-$th axis by angle α. In the classical 'Lieske' parametrization the precession angles z_A, θ_A, ζ_A can be easily expressed in terms of direction cosines X_A, Y_A, V_A, W_A. Quite naturally, all these methods should theoretically lead to the same result.

5. ESTIMATION OF MODEL ACCURACY, COMPARISON WITH OTHER MODELS

In Vondrák et al. (2011b) the accuracy was estimated using a simple expression based on the average uncertainty of all parameters (derived from the fit to integrated values) and weights at different epochs. The uncertainty at epoch T was computed as $\sigma_\circ/w(T)$, where $\sigma_\circ = 0.365''$ was the average unit-weight uncertainty estimated from the fit of all precession parameters, and $w(T)$ the weight defined in Section 2.

Here a rigorous formula is used, based on the full variance-covariance matrix. Thus all correlations existing between estimated parameters are taken into account. For each of the parameters we first estimate the unit-weight uncertainty $\sigma_\circ$ (from the fit

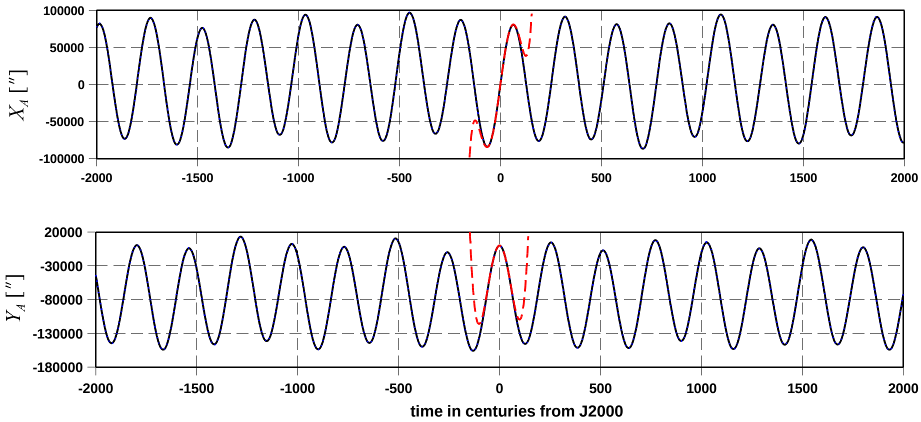

Figure 5: Long-term model of precession parameters X_A, Y_A – new model (dotted), integrated values (solid), and IAU 2006 (dashed).

to numerically integrated values) and then the uncertainty at the epoch T as

$$\sigma^2(T) = \sigma_\circ^2 \sum_{i=1}^{n+4} \sum_{j=1}^{n+4} f_i f_j Q_{ij}, \tag{4}$$

where $f_1 = 1$, $f_2 = T$, $f_3 = T^2$, $f_4 = T^3$, $f_5 = \cos(2\pi T/P_1)$, $f_6 = \sin(2\pi T/P_1)$ ..., and Q_{ij} is the element of the matrix inverse to the matrix of normal equations. The result is depicted in Fig. 6, where the accuracy of each estimated parameter is given and compared with the one from Vondrák et al. (2011b).

It is clear from the figure that our previous estimate was too conservative – the rigorous estimate yields much smaller uncertainties for all parameters, in some cases as much as two orders of magnitude lower.

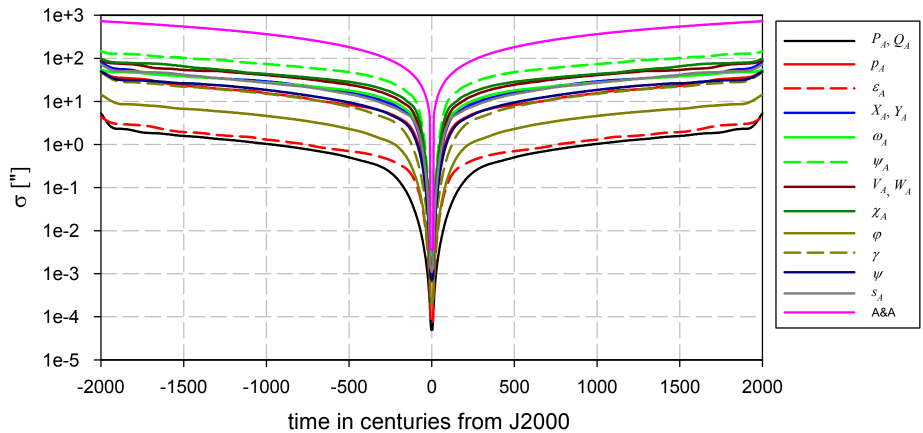

Figure 6: Estimated accuracy of all precession parameters.

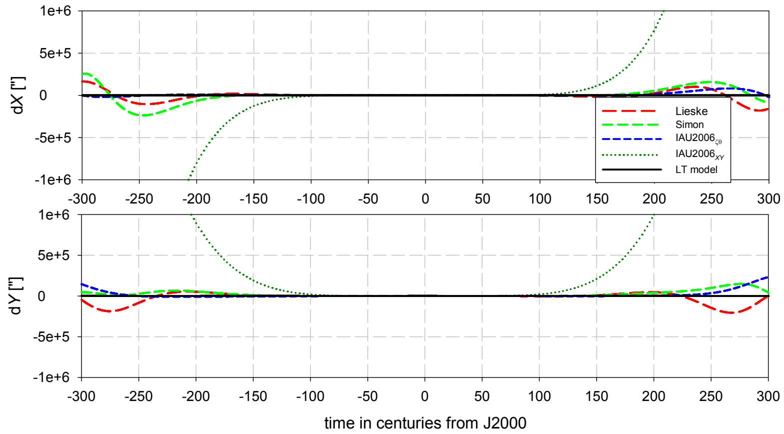

Figure 7: Comparison of different precession models with integrated values.

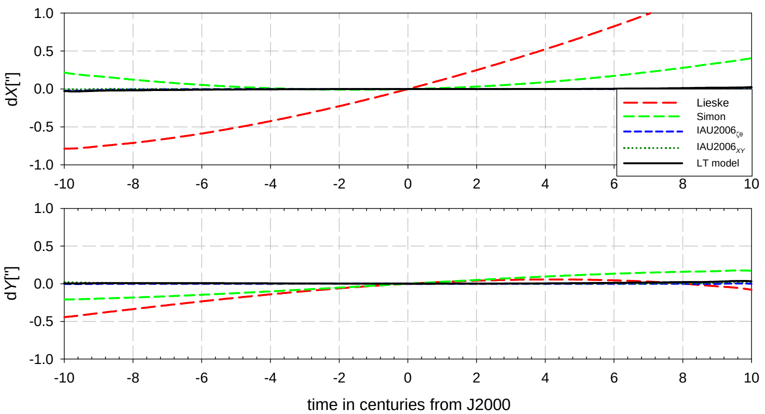

Figure 8: Comparison of precession models – closeup of the central part.

The comparison of the new long-term solution with other models of precession (X_A and Y_A parameters only) is given in Figs 7 and 8. X_A and Y_A values as computed from the values of ζ_A, θ_A by Lieske et al.(1977), Simon et al. (1994) and Capitaine et al. (2003) (denoted as Lieske, Simon, IAU2006$_{\zeta\theta}$), computed directly from the X_A, Y_A expressions by Capitaine et al. (2003), denoted as IAU2006$_{XY}$, and by Vondrák et al. (2011b), denoted as LT model, are compared with the numerically integrated values.

Fig. 7 depicts the comparison in the interval ±300 centuries from J2000.0, while Fig. 8 shows close-up of the central part (±10 centuries from J2000.0). One can see that the direct IAU 2006 expressions for direction cosines X_A, Y_A yield much worse results for more distant epochs than using the expressions of 'traditional' precession angles ζ_A, θ_A. The new LT model is indistinguishable from the integration at this scale, whereas all other models display deviations reaching 50 degrees for epochs more distant than 200 centuries. Fig. 8 clearly demonstrates the correction of precession rate, and also the quadratic term in obliquity, that were recently introduced in more recent models, with respect to Lieske et al. (1977). On the other hand, all models shown are consistent with the numerically integrated precession within one arcsecond or so in the interval ±10 centuries from J2000.0.

6. CONCLUSIONS

The presently adopted IAU 2006 model provides high accuracy over a few centuries around the epoch J2000.0. For longer periods, polynomial development of precession angles ζ_A, θ_A should be preferable to direct X_A, Y_A expressions. More than five thousand years from the fundamental epoch J2000.0 the model IAU 2006 rapidly goes away from reality. The newly proposed model of precession, developed by Vondrák et al. (2011b) and valid over ±200 millennia, is presented. Its accuracy is comparable to IAU 2006 model in the interval of several centuries around J2000.0, and it fits the numerically integrated position of the pole for longer intervals, with gradually decreasing accuracy (several arcminutes ±200 thousand years away from J2000.0). The estimated accuracy, as given in paper (Vondrák et al. 2011b), is too conservative. It is necessary to add that the new model is strictly valid only in the interval ±200 millenia from J2000.0. Outside this interval, its uncertainties rapidly grow, due to strong correlations between the estimated sine/cosine amplitudes of different terms.

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