Asymptotic solution for expanding universe with matter-dominated evolution

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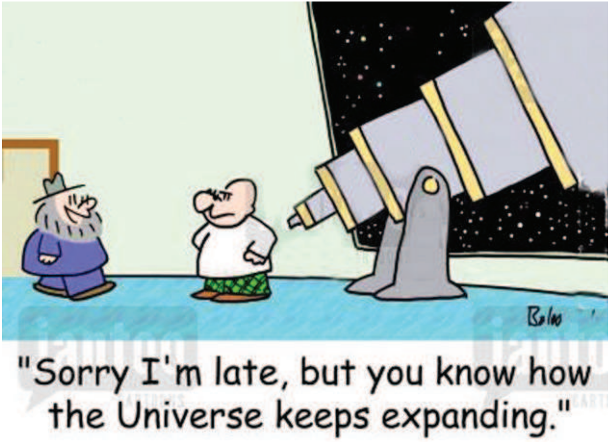


Figure: Excuse of Cosmologist

Friedmann Equations - 1

In our papers [2012], we applied the theory of regularly varying functions in asymptotic analysis of cosmological parameters for the expanding universe. For this analysis we used differential equations, also known as **Friedmann equations**, which are derived from the Einstein field equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \qquad \text{Friedmann equation,}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right), \qquad \text{Acceleration equation,}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, \qquad \text{Fluid equation.}$$

Friedmann equations - 2

Cosmological parameters appearing in these equations are:

$a = a(t)$, the scale factor,
$\rho = \rho(t)$, the energy density, and
$p = p(t)$, the pressure of the material in the universe.

It appears that these parameters, including

$H(t)$, Hubble parameter, and
$q(t)$, deceleration parameter,

are regularly varying functions.

Small digression: definition of a cosmologist

A person who is studying the function 6-tuple

$$(a(t), H(t), q(t), \rho(t), p(t), \Omega(t))$$

but think on galaxies, black holes and Big Bang!

Figure: Twin black holes

Friedmann equations - 3

Representation theory for regularly varying functions says that all these parameters depend on a 0-function $\varepsilon(t)$ which is "hidden" in the integral representation of regularly varying functions.

While $a(t)$, $\rho(t)$ and $H(t)$ uniformly depend only on $\varepsilon(t)$, the parameters $p(t)$ and $q(t)$ depend also on $\dot{\varepsilon}$, what may lead to various evolutions of these parameters.

Continuing our exploration of cosmological parameters from the point of view of regular variations, we derived in our paper [2015] a (Riccati) differential equation for $\varepsilon(t)$

$$t\dot{\varepsilon} = (1-2\alpha)\varepsilon - \varepsilon^2 + \alpha(1-\alpha) - \mu(t),$$

where α is a solution of $x^2 - x + \Gamma = 0$.

Friedmann equations - 4

Here, $\mu(t)$ is the term appearing in the acceleration equation

$$\ddot{a}(t) + \frac{\mu(t)}{t^2} a(t) = 0. \tag{1.1}$$

Hence

$$\mu(t) = \frac{4\pi G t^2}{3} \left(\rho + \frac{3p}{c^2} \right). \tag{1.2}$$

The constant Γ is the integral limit

$$\Gamma = \lim_{x \to \infty} x \int_x^\infty \frac{\mu(t)}{t^2} dt. \tag{1.3}$$

In fact, $\Gamma(\mu) = \mathbf{M}(\mu)$ where $\mathbf{M} \colon \mathfrak{M} \to R$ is a linear functional and $\mathfrak{M}$ is the space of all real functions satisfying the above integral condition (1.3).

Regularly variation

We review the **basic notions related to the regular variation** In particular, we shall need properties of regularly varying solutions of the second order differential equation

$$\ddot{y} + f(t)y = 0, \quad f(t) \text{ is continuous on } [\alpha, \infty]. \tag{2.1}$$

Observe that the acceleration equation has the form (2.1).
In short, the notion of a regular variation is related to the power law distributions, described by the following relationship between quantities F and t:

$$F(t) = t^r(\alpha + o(1)), \quad \alpha, r \in \mathbb{R}. \tag{2.2}$$

Obviously, the most simple form of the power law is given by the equation $y = t^k$.

Regularly varying functions - definition

A real positive continuous function $L(t)$ defined for $x > x_0$ which satisfies

$$\frac{L(\lambda t)}{L(t)} \to 1 \quad \text{as} \quad t \to \infty, \quad \text{for each real } \lambda > 0. \tag{2.3}$$

is called a **slowly varying function**.

A physical quantity F(t) is said to satisfy **the generalized power law** if

$$F(t) = t^r L(t) \tag{2.4}$$

where $L(t)$ is a slowly varying function and r is a real constant.

Examples of regularly varying functions

Examples of slowly varying functions are

$\ln(x)$ and iterated logarithmic functions $\ln(\ldots \ln(x) \ldots)$.

More complicated examples are provided by:

$$L_1(x) = \frac{1}{x}\int_a^x \frac{dt}{\ln t}, \quad L_2(x) = \exp((\ln x)^{1/3} \cos(\ln x)^{1/3}) \quad (2.5)$$

We note that $L_2(x)$ varies infinitely between 0 and ∞.

A positive continuous function F defined for $t > t_0$, is the regularly varying function of the index r, if and only if it satisfies

$$\frac{F(\lambda t)}{F(t)} \to \lambda^r \quad \text{as} \quad t \to \infty, \quad \text{for each } \lambda > 0. \quad (2.6)$$

Regularly varying functions

It immediately follows that a regularly varying function $F(t)$ has the form (2.4).
So to say that $F(t)$ is regularly varying is the same as $F(t)$ to satisfy the generalized power law. If a function $F(x)$ is asymptotically equivalent to a regularly varying function, *it is* a regularly varying function. Hence, we may define the generalized power law also by

$$F(x) \sim t^{\alpha} L(t), \quad \text{as } t \to \infty. \tag{2.7}$$

The class of regularly varying functions of index α we shall denote by $\mathcal{R}_\alpha$. Hence $\mathcal{R}_0$ is the class of all slowly varying functions. By $\mathcal{Z}_0$ we shall denote the class of zero functions at ∞, i.e. $\varepsilon \in \mathcal{Z}_0$ if and only if $\lim\limits_{t\to+\infty} \varepsilon(t) = 0$.

Regularly varying functions

J. Karamata introduced in 1930 the concept of regularly varying functions continuing the works of G.H. Hardy, J.L. Littlewood and E. Landau in the asymptotic analysis of real functions.
The following theorem describes the fundamental property of this class of functions.
Representation theorem $L \in \mathcal{R}_0$ if and only if there are measurable functions $h(x)$ and $\varepsilon \in \mathcal{Z}_0$ and $b \in \mathbb{R}$ so that

$$L(x) = h(x) e^{\int_b^x \frac{\varepsilon(t)}{t} dt}, \quad x \geqslant b, \tag{2.8}$$

and $h(x) \to h_0$ as $x \to \infty$, h_0 is a positive constant.

For our purpose we may assume that $h(x)$ is normalized, i..e that it is a constant h_0.

Density parameter Ω

S.M. Carroll, W.H. Press and E.L. Turner developed in a paper (1992) formulas of the form $H(t)t = F(\Omega)$ for the age of the Universe if the pressureless, i.e. matter dominated Universe is assumed. Here, $\Omega = \Omega(t) = \rho(t)/\rho_c(t)$ is a density parameter. These formulas are also valid for non-zero cosmological constant Λ.

$F(\Omega)$ for flat Universe (k= 0):

$$F(\Omega) = \frac{2}{3}(1-\Omega)^{-\frac{1}{2}} \ln\left(\frac{1+\sqrt{1-\Omega}}{\sqrt{\Omega}}\right) \tag{2.9}$$

$F(\Omega)$ for open Universe (k=-1):

$$F(\Omega) = \frac{1}{1-\Omega} - \frac{\Omega}{2(1-\Omega)^{\frac{3}{2}}} \cosh^{-1}\left(\frac{2-\Omega}{\Omega}\right) \tag{2.10}$$

Density parameter Ω

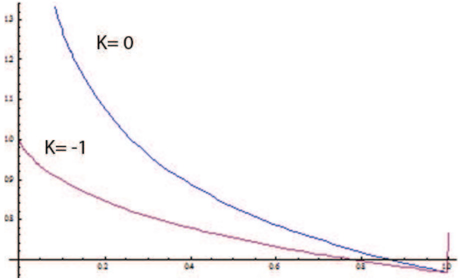


Figure: Graph of $F(\Omega)$

Ω and regular variation

From now on we shall assume matter dominated Universe with cosmological constant Λ. We show how to infer asymptotic formulas for the scale factor $a(t)$ using $F(\Omega)$.

For this, we note that the Hubble parameter is the logarithmic derivative of $a(t)$:

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \frac{d \ln\, a(t)}{dt}.$$

Hence

$$a(t) = e^{\int H(t)dt} = e^{\int \frac{H(t)t}{t}dt} = e^{\int \frac{\Omega(t)}{t}dt}. \tag{2.11}$$

Ω and regular variation

Case: There is a limit $\lim_{t\to\infty} \Omega(t) = \alpha > 0$.

Then

$$\frac{a(t)}{a(t_0)} = \exp\left(\int_{t_0}^{t} \frac{\Omega(t)}{t} dt\right)$$
$$= \exp\left(\int_{t_0}^{t} \frac{\alpha}{t} dt + \int_{t_0}^{t} \left(\frac{\Omega(t)-\alpha}{t}\right) dt\right).$$

Hence

$$a(t) = a(t_0)\left(\frac{t}{t_0}\right)^{\alpha} L_0(t), \quad t > t_0 > 0. \qquad (2.12)$$

where $L_0(t) = e^{\int_{t_0}^{t} \frac{\varepsilon(t)}{t} dt}$, $\varepsilon(t) = \frac{\Omega(t)-\alpha}{t} \to 0$ if $t \to \infty$.

Hence, $L_0(t)$ is slowly regular, and so $a(t)$ is regular.

Ω and regular variation

Note.

Functions $F(\Omega)$ are defined for $0 < \Omega < 1$. If $\Omega \sim 1$, then:

$$F(\Omega) = \frac{2}{3}\left(1 + \frac{\sqrt{1-\Omega}}{1+\sqrt{\Omega}}\right) + o(1-\Omega), \quad (k=0).$$

$$F(\Omega) = \frac{2}{\Omega} - \frac{4}{3\Omega^2} + \frac{2\sqrt{1-\Omega}}{\Omega} + o(1-\Omega), \quad (k=-1).$$

Hence, in both cases $\lim_{\Omega\to 1} F(\Omega) = 2/3$ as $\Omega \to 1$. It is also easy to that

$$\lim_{\Omega\to 0} F(\Omega) = +\infty \quad \text{as} \quad \Omega \to 0 \quad (k=0).$$

$$\lim_{\Omega\to 0} F(\Omega) = 1 \quad \text{as} \quad \Omega \to 0 \quad (k=-1).$$

Ω and regular variation

Therefore, if $\lim_{t\to\infty}\Omega(t)$ exists (and is equal to $\alpha > 0$), then the scale factor $a(t)$ satisfies **generalized power law**:

$a(t) = t^{\alpha}L_0(t)$, $L_0(t)$ is slowly varying.

According to [Mijajlovic at all, 2012] then all cosmological parameters are uniquely determined and a form of equation of state holds.

We shall present formulas for these parameters only for curvature index $k = 0$.

Ω and regular variation, k=0

First we introduce the related constants following their definition in our paper (2012).

We remaind that the threshold constant Γ is in relation to acceleration equation and it is defined by (1.3). Since α is a solution of $x^2 - x + \Gamma = 0$, we take

$$\Gamma = \alpha(1-\alpha).$$

For curvature index $k = 0, -1$ it must be $\Gamma < 1/4$. This condition is fulfilled since $\alpha < 2/3$.

The equation of state constant w is defined by

$$w = \frac{2}{3\alpha} - 1. \tag{2.13}$$

Ω and regular variation, k=0

Cosmological parameters:

$$\begin{aligned} \alpha &= \frac{2}{3(1+w)}, & a(t) &= a_0 t^{\frac{2}{3(1+w)}} L(t) \\ H(t) &\sim \frac{2}{3(1+w)t}, & \mathbf{M}(q) &= \frac{1+3w}{2} \end{aligned} \tag{2.14}$$

Also, there are 0-functions ξ, ζ such that

$$p = \hat{w}\rho c^2, \quad \text{(equation of state)}$$

where $\hat{w}(t) = w - t\dot{\xi} + \zeta$.

Ω and acceleration equation

Acceleration equation with cosmological parameter Λ is written as follows:

$$\frac{\ddot{a}}{a} = -\frac{\mu}{t^2} + \frac{\Lambda}{3}$$

Deceleration parameter $q(t)$ is defined by $q = -\frac{\ddot{a}}{aH^2}$. Hence

$$q = \frac{\mu}{(tH)^2} - \frac{\Lambda}{3H^2} \quad \text{so} \quad q = \frac{\mu}{F(\Omega)^2} - \Omega_\Lambda$$

As $q = \Omega/2 - \Omega_\Lambda$ we obtain at once

$$\mu = \frac{\Omega}{2} F(\Omega)^2.$$

Here, Ω_Λ is so called a density parameter of the cosmological constant.

Ω and acceleration equation

Friedman equations with cosmological constant Λ can be simplified by substitutions

$$\rho_\Lambda = \rho - \frac{\Lambda c^2}{8\pi G}, \quad p_\Lambda = p + \frac{\Lambda c^4}{8\pi G}$$

Then the acceleration equation is reduced to the standard form

$$\frac{\ddot{a}}{a} = -\frac{\mu_\Lambda}{t^2}, \quad \text{or} \quad q = \frac{\mu_\Lambda}{F(\Omega)^2}.$$

As $q = \Omega/2 - \Omega_\Lambda$ and in case of $k = 0$ the identity $\Omega + \Omega_\Lambda = 1$ holds, we at once obtain

$$\mu_\Lambda = (\Omega/2 - \Omega_\Lambda)F(\Omega)^2, \quad \mu_\Lambda = (3\Omega/2 - 1)F(\Omega)^2.$$

Ω and acceleration equation

The limit value

$$\Omega_\infty = \lim_{t\to\infty} \Omega(t)$$

can be in principle any value in the interval $[0,1]$. Let $\bar{\mu}(\Omega)$ be a function of Ω given by the expression μ_Λ. Thus $\mu_\Lambda(t) = \bar{\mu}(\Omega(t))$ and by our paper [2012] the threshold constant Γ is

$$\Gamma = \lim_{t\to\infty} \mu_\Lambda(t) = \lim_{\Omega\to\Omega_\infty} \bar{\mu}(\Omega).$$

We see that $\bar{\mu}(\Omega)$ is an increasing function in Ω and that its values lay in the interval $[-\infty, \frac{2}{9}]$, as $\lim\limits_{\Omega\to 1-0} \bar{\mu}(\Omega) = \dfrac{2}{9}$. Hence $\mu(t) < \dfrac{2}{9}$. Then $\Gamma \leqslant \frac{2}{9} < \frac{1}{4}$. Thus, assuming the pressureless spatially flat universe with the cosmological constant, we again obtain that the scale parameter $a(t)$ satisfies the power law. The difference to the previous deduction is that Γ is computed firstly then α as a solution of $x^2 - x + \Gamma = 0$.

Ω and acceleration equation

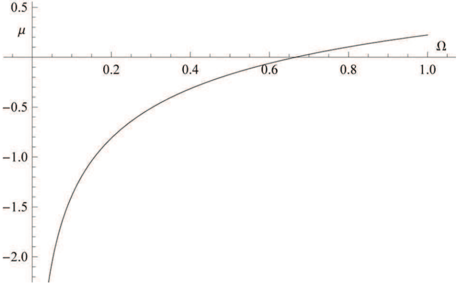


Figure: Graph of $\bar{\mu}(\Omega)$

Ω and acceleration equation

Case: There is no limit $\lim_{t\to\infty}\Omega(t)$. In this case one can show that there are real numbers m, n, $m < n$ and

$$t^m < a(t) < t^n, \quad t > t_0.$$

In fact one can show that there is a function $u(t)$ such that

$$a(t) = t^m \cos(u(t))^2 + t^n \sin(u(t))^2.$$

Also, $a(t)$ abuts infinitely many times both functions t^m and t^n. The graph of $a(t)$ is similar to the displayed one.

Ω and acceleration equation

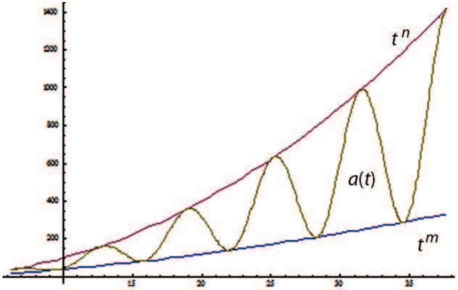
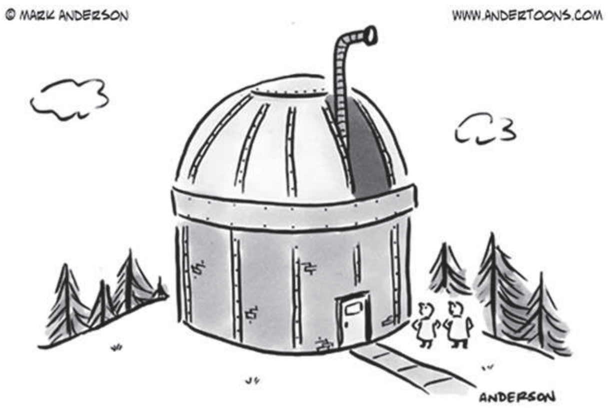


Figure: Graph of $a(t)$) if $\lim_{t\to\infty} \Omega(t)$ does not exist

Telescope

They ordered telescope, but...



"Hmm... Lemme check that purchase order again."

References

Keywords: Cosmological parameters, Regular variations

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