NONSINGULAR BIG BANG IN NONLOCAL MODIFIED GRAVITY

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Motivation

Large cosmological observational findings:

- High orbital speeds of galaxies in clusters. (F.Zwicky, 1933)
- High orbital speeds of stars in spiral galaxies. (Vera Rubin, at the end of 1960es)
- Accelerated expansion of the Universe. (1998)

Problem solving approaches

There are two problem solving approaches:

- Dark matter and energy
- Modification of Einstein theory of gravity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},\ c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.

Dark matter and energy

- If Einstein theory of gravity can be applied to the whole Universe then the Universe contains about 4.9% of ordinary matter, 26.8% of dark matter and 68.3% of dark energy.
- It means that 95.1% of total matter, or energy, represents dark side of the Universe, which nature is unknown.
- Dark matter is responsible for orbital speeds in galaxies, and dark energy is responsible for accelerated expansion of the Universe.

Modification of Einstein theory of gravity

Motivation for modification of Einstein theory of gravity

- The validity of General Relativity on cosmological scale is not confirmed.
- Dark matter and dark energy are not yet detected in the laboratory experiments.
- Another cosmological problem is related to the Big Bang singularity. Namely, under rather general conditions, general relativity yields cosmological solutions with zero size of the universe at its beginning, what means an infinite matter density.
- Note that when physical theory contains singularity, it is not valid in the vicinity of singularity and must be appropriately modified.

Approaches to modification of Einstein theory of gravity

There are different approaches to modification of Einstein theory of gravity.

- Einstein General Theory of Relativity

From action $S = \int (\frac{R}{16\pi G} - L_m - 2\Lambda)\sqrt{-g}d^4x$ using variational methods we get field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu},\ c = 1$$

where $T_{\mu\nu}$ is stress-energy tensor, $g_{\mu\nu}$ are the elements of the metric tensor, $R_{\mu\nu}$ is Ricci tensor and R is scalar curvature of metric.
Currently there are mainly two approaches:

- f(R) Modified Gravity
- Nonlocal Gravity

Nonlocal Modified Gravity

Nonlocal gravity is a modification of Einstein general relativity in such way that Einstein-Hilbert action contains a function f(□, R). Our action is given by

$$S = \int d^4x\sqrt{-g}\Big(\frac{R - 2\Lambda}{16\pi G} + R^p\mathcal{F}(\Box)R^q\Big)$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu$, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n\Box^n$.

We use Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\big),\ k \in \{-1, 0, 1\}.$$

Equations of motion

Equation of motion are

$$-\frac{1}{2}g_{\mu\nu}R^p\mathcal{F}(\Box)R^q + R_{\mu\nu}W - K_{\mu\nu}W + \frac{1}{2}\Omega_{\mu\nu} = -\frac{G_{\mu\nu}+\Lambda g_{\mu\nu}}{16\pi G},$$

$$\Omega_{\mu\nu} = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(g_{\mu\nu}\nabla^\alpha\Box^l R^p \nabla_\alpha \Box^{n-1-l}R^q$$

$$-2\nabla_\mu\Box^l R^p\nabla_\nu\Box^{n-1-l}R^q + g_{\mu\nu}\Box^l R^p\Box^{n-l}R^q\big),$$

$$K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box,$$

$$W = pR^{p-1}\mathcal{F}(\Box)R^q + qR^{q-1}\mathcal{F}(\Box)R^p.$$

Trace and 00-equations

In case of *FRW* metric there are two linearly independent equations. The most convenient choice is trace and 00 equations:

$$
\begin{aligned}
&-2R^p\mathcal{F}(\Box)R^q + RW + 3\Box W + \frac{1}{2}\Omega = \frac{R-4\Lambda}{16\pi G},\\
&\frac{1}{2}R^p\mathcal{F}(\Box)R^q + R_{00}W - K_{00}W + \frac{1}{2}\Omega_{00} = -\frac{G_{00}-\Lambda}{16\pi G},\\
&\Omega = g^{\mu\nu}\Omega_{\mu\nu}.
\end{aligned}
$$

Cosmological solutions in nonlocal modified gravity

We discuss the class of models given by the action

$$S = \int_M \left(\frac{R - 2\Lambda}{16\pi G} + R^p \mathcal{F}(\Box) R^q \right) \sqrt{-g}\ d^4x.$$

In particular, we investigate the following cases:

- $p \in \mathbb{N}$, $q \in \mathbb{N}$
- $p = 1$, $q = 1$,
- $p = -1$, $q = 1$,

First case $p \in \mathbb{N}$, $q \in \mathbb{N}$

We investigate the model

$$S = \int_M \left(\frac{R - 2\Lambda}{16\pi G} + R^p \mathcal{F}(\Box) R^q \right) \sqrt{-g}\; d^4 x.$$

Also, we assume $p \geq q$.

First case $p \in \mathbb{N}$, $q \in \mathbb{N}$

We discuss only $k = 0$. Scale factor is chosen in the form

$$a(t) = a_0 e^{-\frac{\gamma}{12} t^2}, \quad \gamma \in \mathbb{R}.$$

Hubble parameter and scalar curvature are given by

$$H(t) = -\frac{1}{6}\gamma t, \qquad R(t) = \frac{1}{3}\gamma(\gamma t^2 - 3), \qquad R_{00} = \frac{1}{4}(\gamma - R).$$

First case $p \in \mathbb{N}$, $q \in \mathbb{N}$

$$\Box R^p = p\gamma R^p - \frac{p}{3}(4p-5)\gamma^2 R^{p-1} - \frac{4}{3}p(p-1)\gamma^3 R^{p-2}.$$

From this relation it follows that operator $\Box$ is closed on the space of polynomials in R of degree at most p. In the basis
$v_p = \begin{pmatrix} R^p & R^{p-1} & \dots & R & 1 \end{pmatrix}^T$ the matrix of $\Box$ is

$$M_p = \gamma \begin{pmatrix} p & \frac{p}{3}(5-4p)\gamma & \frac{4}{3}p(1-p)\gamma^2 & 0 & \dots & 0 \\ 0 & p-1 & \frac{p-1}{3}(9-4p)\gamma & \frac{4}{3}(1-p)(p-2)\gamma^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & \frac{\gamma}{3} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

EOM

Moreover, $F_p = \sum_{n=0}^{\infty} f_n M_p^n$ is a matrix of operator $\mathcal{F}(\Box)$. Let D_p be a matrix of operator $\frac{\partial}{\partial R}$ and e_p are the coordinates of vector R^p in basis v_p.

Trace and 00 equation are transformed into

$$T = 0, \qquad Z = 0,$$

where

$$\begin{aligned} T &= -2e_p v_p e_q F_q v_q + RW_{pq} - 4\gamma^2(R+\gamma)W''_{pq} - 2\gamma^2 W'_{pq} \\ &- S_1 + 2S_2 - \frac{R-4\Lambda}{16\pi G}, \\ Z &= \frac{1}{2}e_p v_p e_q F_q v_q + \frac{\gamma}{4}(\gamma - R)W_{pq} - \gamma(R+\gamma)W'_{pq} \\ &- \frac{1}{2}(S_1 + S_2) + \frac{G_{00} - \Lambda}{16\pi G}, \end{aligned}$$

EOM

and

$$S_1 = \frac{4}{3}\gamma^2(R+\gamma)\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} e_p M_p^l D_p v_p e_q M_q^{n-1-l} D_q v_q,$$

$$S_2 = \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} e_p M_p^l v_p e_q M_q^{n-l} v_q.$$

EOM

Theorem

For all $p \in \mathbb{N}$ and $q \in \mathbb{N}$ we have $T + 4Z = 4\gamma Z'$. Trace and 00 equations are equivalent.

Therefore, it is sufficient to solve only trace equation.
It is of a polynomial type, degree $p+q$ in R, and it splits into $p+q+1$ equations with $p+q+1$ "variables" $f_0 = \mathcal{F}(0)$, $\mathcal{F}(\gamma)$, $\dots$, $\mathcal{F}(p\gamma)$, $\mathcal{F}'(\gamma)$, $\dots$, $\mathcal{F}'(q\gamma)$.

$(p, q) = (1, 1)$

Theorem

For $p = q = 1$, *trace equation is satisfied iff* $\gamma = -12\Lambda$, $\mathcal{F}'(\gamma) = 0$ *and* $f_0 = \frac{3}{32\gamma\pi G} - 8\mathcal{F}(\gamma)$.

$(p,q) \neq (1,1)$

Theorem

Trace equation is satisfied for the following values of parameters p and q ($\kappa = \frac{1}{16\pi G}$):

- $p=2,\ q=1$: $\mathcal{F}(\gamma) = \frac{9\kappa(\gamma+9\Lambda)}{112\gamma^3}$, $\mathcal{F}(2\gamma) = \frac{3\kappa(\gamma+9\Lambda)}{56\gamma^3}$, $f_0 = -\frac{\kappa(4\gamma+15\Lambda)}{7\gamma^3}$, $\mathcal{F}'(\gamma) = -\frac{3\kappa(\gamma+9\Lambda)}{8\gamma^4}$,
- $p=2,\ q=2$: $\mathcal{F}(\gamma) = \frac{369\kappa(\gamma+8\Lambda)}{9344\gamma^4}$, $\mathcal{F}(2\gamma) = \frac{27\kappa(\gamma+8\Lambda)}{4672\gamma^4}$, $f_0 = \frac{\kappa(145\gamma+576\Lambda)}{876\gamma^4}$, $\mathcal{F}'(\gamma) = -\frac{639\kappa(\gamma+8\Lambda)}{2336\gamma^5}$, $\mathcal{F}'(2\gamma) = -\frac{27\kappa(\gamma+8\Lambda)}{9344\gamma^5}$,
- $p=3,\ q=1$: $\mathcal{F}(\gamma) = \frac{\kappa(107\gamma+408\Lambda)}{6432\gamma^4}$, $\mathcal{F}(2\gamma) = -\frac{\kappa(173\gamma+840\Lambda)}{7504\gamma^4}$, $\mathcal{F}(3\gamma) = 0$, $f_0 = -\frac{\kappa(95\gamma+768\Lambda)}{268\gamma^4}$, $\mathcal{F}'(\gamma) = -\frac{9\kappa(\gamma+8\Lambda)}{88\gamma^5}$.
- $p=3,\ q=2$: $\mathcal{F}(\gamma) = \frac{3\kappa(10702\gamma+40497\Lambda)}{245680\gamma^5}$, $\mathcal{F}(2\gamma) = -\frac{27\kappa(6\gamma+25\Lambda)}{24568\gamma^5}$, $\mathcal{F}(3\gamma) = -\frac{27\kappa(6\gamma+25\Lambda)}{49136\gamma^5}$, $f_0 = -\frac{3\kappa(7099\gamma+23949\Lambda)}{15355\gamma^5}$, $\mathcal{F}'(\gamma) = -\frac{3\kappa(11614\gamma+68865\Lambda)}{270248\gamma^6}$, $\mathcal{F}'(2\gamma) = \frac{513\kappa(6\gamma+25\Lambda)}{171976\gamma^6}$,

Summary, $p \in \mathbb{N}$, $q \in \mathbb{N}$

- We discuss the scale factor of the form $a(t) = a_0 \exp(-\frac{\gamma}{12} t^2)$
- Trace and 00 equations are equivalent. Trace equation is of a degree $p+q$ in R, so it splits into $p+q+1$ equations over $p+q+1$ "variables" $f_0 = \mathcal{F}(0)$, $\mathcal{F}(\gamma)$, $\dots$, $\mathcal{F}(p\gamma)$, $\mathcal{F}'(\gamma)$, $\dots$, $\mathcal{F}'(q\gamma)$.
- For $p = q = 1$ the system has infinitely many solutions, and constants γ and Λ satisfy $\gamma = -12\Lambda$.
- For other values of p and q, there is unique solution, for any $\gamma \in \mathbb{R}$.
- We obtained solutions for $1 \leq q \leq p \leq 4$.

Second case $p = 1$, $q = 1$

For $p = q = 1$ the action becomes

$$S = \int_M \left(\frac{R - 2\Lambda}{16\pi G} + R\mathcal{F}(\Box)R \right) \sqrt{-g}\, d^4x.$$

This model is interesting since we obtain nonsingular cosmological solutions. To solve EOM we use the ansatz:

$\Box R = rR + s, \qquad r, s \in \mathbb{R}$

Nonsingular bounce cosmological solutions

Let the scale factor $a(t)$ be in a form

$$a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t}), \quad a_0 > 0, \; \lambda, \sigma, \tau \in \mathbb{R}.$$

Theorem

Scale factor $a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t})$ is a solution of EOM in a following three cases:

Case 1.

$$\mathcal{F}\left(2\lambda^2\right) = 0, \quad \mathcal{F}'\left(2\lambda^2\right) = 0, \quad f_0 = -\frac{1}{128\pi G C \Lambda}.$$

Case 2. $3k = 4a_0^2 \Lambda \sigma \tau$.

Case 3.

$$\mathcal{F}\left(2\lambda^2\right) = \frac{1}{192\pi G C \Lambda} + \frac{2}{3} f_0, \quad \mathcal{F}'\left(2\lambda^2\right) = 0, \quad k = -4a_0^2 \Lambda \sigma \tau.$$

In all cases we have $3\lambda^2 = \Lambda$.

Summary, $p = 1$, $q = 1$

- We studied nonlocal gravity model with cosmological constant Λ, without matter.
- Using anzac $\Box R = rR + s$ we found three types of nonsingular bounce cosmogical solutions with scale factor $a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t})$.
- All solutions satisfy

$$\ddot{a}(t) = \lambda^2 a(t) > 0.$$

- There are solutions for all values of parameter $k = 0, \pm 1$.

Third case $p = -1$, $q = 1$

This model is given by the action

$$S = \int_M \left(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R \right) \sqrt{-g}\; d^4x,$$

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$. If we set $f_0 = -\frac{\Lambda}{8\pi G}$, then f_0 takes the place of cosmological constant.

Third case $p = -1$, $q = 1$

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Some power-law cosmological solutions

We use the following form of scale factor and scalar curvature:

$$a(t) = a_0|t - t_0|^{\alpha},$$
$$R(t) = 6\Big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\Big).$$

Case $k = 0$, $\alpha \neq 0$ i $\alpha \neq \frac{1}{2}$

Theorem

For $k = 0$, $\alpha \neq 0$, $\alpha \neq \frac{1}{2}$ and $\frac{3\alpha-1}{2} \in \mathbb{N}$ scale factor $a = a_0|t - t_0|^\alpha$ is a solution of EOM if

$$
\begin{aligned}
& f_0 = 0,\ f_1 = -\frac{3\alpha(2\alpha-1)}{32\pi G(3\alpha-2)}, \\
& f_n = 0 \quad za \quad 2 \leq n \leq \frac{3\alpha-1}{2}, \\
& f_n \in \mathbb{R} \quad za \quad n > \frac{3\alpha-1}{2}.
\end{aligned}
$$

Case $k = 0$, $\alpha \to 0$ (Minkowski spacetime)

Theorem

For $k = 0$ and $\alpha \to 0$ EOM are satisfied if

$$f_0, f_1 \in \mathbb{R}, \quad f_i = 0, \ i \geq 2.$$

Since all $f_n = 0$, $n \geq 2$ this model is not a nonlocal model. Thus the power-law solutions cannot be obtained by perturbing the Minkowski solution.

Case $k = 0$, $\alpha \to \frac{1}{2}$

Theorem

For $k = 0$ and $\alpha \to \frac{1}{2}$ EOM are satisfied if

$$f_0 \in \mathbb{R}, \quad f_i = 0, \ i \geq 1.$$

Case $k \neq 0$, $\alpha = 1$

Let $k \neq 0$. To simplify the scalar curvature

$$R(t) = 6\big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\big)$$

we have three options: $\alpha = 0$, $\alpha = \frac{1}{2}$ i $\alpha = 1$. The first two does not yield any solutions of EOM and the last one is described in the following theorem.

Theorem

For $k \neq 0$ scale factor $a = a_0|t - t_0|$ is a solution of EOM if

$$f_0 = 0, \quad f_1 = \frac{-s}{64\pi G}, \quad f_n \in \mathbb{R}, \quad n \geq 2,$$

where $s = 6(1 + \frac{k}{a_0^2})$.

Summary, $p = -1$, $q = 1$

- We the model with nonlocal term $R^{-1}\mathcal{F}(\Box)R$ and obtained power-law cosmological solutions $a(t) = a_0|t - t_0|^{\alpha}$.
- It is worth noting that there is a solution $a(t) = |t - t_0|$ which corresponds to Milne universe for $k = -1$.
- All presented solutions $a(t) = a_0|t - t_0|^{\alpha}$ have scalar curvature $R(t) = 6\big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\big)$, that satisfies $\Box R = qR^2$, where parameter q depends on α.

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Thank you!